

# SYNCHRONIZATION & COLLISION AVOIDANCE IN NON-LINEAR FLOCKING NETWORKS OF AUTONOMOUS AGENTS [EXTENDED VERSION]

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**ABSTRACT.** *This is an accompanying technical report of a conference paper with the same title, that is to be presented in the 24th Mediterranean Conference on Control and Automation (MED), in Athens Greece, June 21-24, 2016 [22].*

We discuss two extensions of second-order consensus networks with state-dependent couplings of Cucker-Smale type. The first scheme models flocking to synchronization over a network of agents. The alignment of the agent's states occurs over a non-trivial limit orbit that is generated by the internal dynamics of the agents. The second scheme models the velocity alignment of a group of agents which avoid approaching each other closer than a prescribed distance. While seemingly different, both of these systems can be analyzed using the same mathematical methods. We analyze both models and reveal their structural similarities. Sufficient conditions between the initial configurations and the system's network parameters are derived. Simulation examples are presented to illustrate our theoretical conclusions.

**1. Introduction.** The dynamics of networked agents are of immense importance in various disciplines of applied science. In recent years, there has been a broad interest in the study of distributed co-operative dynamic algorithms run among a finite number of agents. The most prominent family of co-operative algorithms is this of the consensus networks (see for example [7, 10, 11, 12, 14, 18, 23] and references therein).

The standard setting of a consensus network regards a number of agents  $N < \infty$  labeled to form a group  $[N] = \{1, \dots, N\}$ . Every member  $i \in [N]$  possesses a value of interest  $z_i \in \mathbb{R}$  that is initialized and updated according to:

$$i \in [N] : \begin{cases} \dot{z}_i(t) = \sum_j w_{ij}(t)(z_j(t) - z_i(t)), & t \geq t_0 \\ z_i(t) = z_i^0, & t = t_0. \end{cases} \quad (1)$$

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The parameters  $w_{ij} \geq 0$ , known as coupling rates, characterize the effect of agent  $j$  onto  $i$ . Certain connectivity criteria (thoroughly discussed in the aforementioned works) ensure that the solution  $\mathbf{z} = (z_1, \dots, z_N)$  satisfies

$$z_i(t) \rightarrow z^* \text{ as } t \rightarrow \infty, \forall i = 1, \dots, N$$

for some value  $z^* \in [\min_i z_i^0, \max_i z_i^0]$ .

The research on stability conditions for (1) is already a saturated subject. On the contrary, there are relatively few non-linear versions [1, 4, 8, 13, 14] and even fewer non-linear schemes that coordinate the agents in order to collectively execute a more complex task, than this of convergence to a common constant [2].

A similar to (1) family of networks is this where each agent is defined through the pair of  $(x_i, u_i) \in \mathbb{R} \times \mathbb{R}$  usually understood as position and velocity and attempts to coordinate its velocity with respect to its neighboring agents. These systems are known as second order (flocking) networks and linear versions have attracted the research interest of the control community [24]. Recent advances regard non-linear variations of (1). In a series of papers [4, 3] a significant non-linear version of second order algorithms is introduced and analyzed:

$$i \in [N] : \begin{cases} \dot{x}_i(t) = u_i(t) \\ \dot{u}_i(t) = \sum_j w_{ij}(x(t)) (u_j(t) - u_i(t)), & t \geq t_0 \\ u_i(t) = u_i^0, \quad x_i(t) = x_i^0, & t = t_0. \end{cases} \quad (2)$$

where  $w_{ij}(x) = \frac{K}{1 + \|x_i - x_j\|^{2\beta}}$  are distance-dependent couplings. The working hypothesis (2) illustrates that the further the relative distance between two agents is, the less effect they have on each other. It then may occur that the agents will not be positioning themselves sufficiently close so that the network preserves the necessary connectivity strength in order for global speed alignment to occur. This is, in fact, the actual challenge in systems like (2): the derivation of appropriate initial conditions so that global flocking emerges. One would expect a relation between the initial data and the coupling parameters  $K$  and  $\beta$ . The system (2) has attracted immense interest over the years and it has been substantially improved in various ways [5, 14, 17, 20].

**Theorem 1.1** ([14, 20]). *Consider the system (2) and its solution  $(\mathbf{x}, \mathbf{u})$  with  $w_{ij}(x) \geq \psi(\max_{i,j} |x_i - x_j|)$  and  $\psi$  an appropriate non-negative integrable function. Then, convergence to a common equilibrium velocity occurs exponentially fast if the initial data satisfy*

$$\max_{i,j} |u_i^0 - u_j^0| < \int_{\max_{i,j} |x_i^0 - x_j^0|}^{\infty} \psi(s) ds.$$

**1.1. Related Work.** In a series of papers the authors have adapted concepts from the theory of non-negative matrices in order to develop a unified study of consensus dynamics in many versions [19, 17, 18, 20, 21]. The main advantage of this approach is that it uses fairly mild assumptions to provide strong stability results with explicit estimates on the rate of convergence. The characteristic notion used in these works is an estimator of the averaging effect of non-negative stochastic matrices, known as the contraction coefficient [6, 15]. This is the fundamental tool used in the central convergence results on the products of non-negative matrices and non-homogeneous Markov Chains. Additionally, it provides the underlying framework for investigating the stability of general linear consensus networks [7, 14, 18].

While in most related works, the rate of convergence is strongly associated with the performance of the network [16]; flocking networks of Cucker-Smale type require rate estimates as a prerequisite to prove asymptotic stability. In [21] the authors used the contraction coefficient for the study of first order, non-linear networks that lie beyond the form of (1). In the present work, we discuss the extension of the contraction coefficient to two types of second-order consensus-based systems.

**1.2. Contribution.** We introduce and discuss two seemingly different examples of networks and we reveal how they can be studied under a fairly similar methodology. The first example is an extension of the classic Cucker-Smale type of networks for bird flocking to a differential equation form of the kind of [21] where the coordination process involves only the velocity variable. In this scenario, the agents are assumed to have an inherent common dynamic rule that governs the update of their velocity in addition to the collective coupling. The results state sufficient conditions on the initial configuration of the position and the velocity in order for the flock to achieve asymptotic alignment around a limit set that need not be an equilibrium.

The second example is a collision avoidance dynamic network. In this scenario, the agents attempt to achieve collective consensus around their velocity with the additional restriction that the agents must be kept away from each other at a minimum distance. The result is a condition on the initial configuration and the system's parameters that guarantees convergence around a common speed while the flock remains connected at all times.

The stability properties of both of these networks are rigorously analyzed with appropriate modification of the mathematics used for examining the averaging properties of the contraction coefficient. This enables us to consider milder connectivity assumptions that extend and generalize existing results [2].

The present report serves as an accompanying technical reference of a conference paper with the same title, that is to be presented in the 24th Mediterranean Conference on Control and Automation (MED), Athens Greece, June, 2016 [22].

**1.3. Organization of the paper.** The rest of the report is organized as follows. In §2 we introduce some notation and we present underlying results that will come of use. In §3 we state the two problems, we make some introductory remarks on the derivation of the particular forms, we state the escorting assumptions on the various parameters involved and conclude with the presentation of the main stability results. The proof of the results is carried out in §4. Simulation examples are presented in §5. Discussion and concluding remarks for future work are provided in §6.

## 2. Preliminaries.

**2.1. Basic Notation.** Let  $N < \infty$  denote the number of autonomous agents and  $[N]$  as defined above. The communication scheme is represented by a weighted graph  $G = ([N], E)$  where  $[N]$  is the set of agents,  $E = \{a_{ij} : i, j \in [N]\}$  is the set of edges. We denote the weighted degree of node  $i$ , as  $d_i = \sum_j a_{ij}$ . The dynamics evolve in  $\mathbb{R}^m$  for some  $m \geq 1$  that is endowed with the inner product  $\langle \cdot, \cdot \rangle$  and the corresponding euclidean norm  $\|\cdot\|$ . For all  $z \in \mathbb{R}^m$  we also define  $|z| = \max_{i=1, \dots, m} |z^i|$ . Each agent  $i \in [N]$  is characterized by the state  $(x_i, u_i) \in \mathbb{R}^m \times \mathbb{R}^m$ . Clearly  $x_i = (x_i^{(1)}, \dots, x_i^{(m)})$  stands for the position of  $i$  and  $u_i = (u_i^{(1)}, \dots, u_i^{(m)})$  stands for its velocity. In compact form we write  $\mathbf{x} = (x_1, \dots, x_N)$ ,  $\mathbf{u} = (u_1, \dots, u_N)$

both elements of the augmented space  $\mathbb{R}^{N \times m}$ . For  $l = 1 \dots, m$ , the spread of  $\mathbf{y} \in \mathbb{R}^{N \times m}$  in the  $l$ -th dimension is

$$S_l(\mathbf{y}) = \max_i y_i^{(l)} - \min_i y_i^{(l)} = \max_{i,j} |y_i^{(l)} - y_j^{(l)}|$$

and finally we denote  $S(\mathbf{y}) = \max_l S_l(\mathbf{y})$ . We also set  $x_{i,j} = x_i - x_j$  and  $u_{i,j} = u_i - u_j$  the difference in terms of vectors, for the sake of convenience. The space of continuous functions defined in  $S$  and taking values in  $V$  with  $s \geq 0$  continuous derivatives is defined as  $C^s(S, V)$ . Throughout this paper any derivative is defined in the extended sense, i.e.  $d/dt$  or “ $\cdot$ ” denotes the right-hand derivative operator.

**2.2. The contraction coefficient.** The development of Markov Chains as well as the main mathematical proof tool in the background of many works on consensus networks is the coefficient of ergodicity. An idea developed independently by various mathematicians but was first introduced in one of Markov’s first papers [6].

**Theorem 2.1.** *Let  $A = [a_{ij}]$  be a non-negative matrix with constant row sums (i.e.  $\sum_j a_{ij} \equiv n$  for some  $n > 0$ ). Then*

$$S(Az) \leq (n - \min_{i,i'} \sum_k \{a_{ik}, a_{i'k}\}) S(z).$$

The proof of Theorem 2.1 can be found in [6]. The contraction coefficient is the quantity  $(n - \min_{i,i'} \sum_k \{a_{ik}, a_{i'k}\})$ . Although the classic theory of non-negative matrices clearly regard non-negative  $a_{ij}$ , the background proofs like this on the contraction coefficient can be used to handle examples of negative values of  $a_{ij}$ .

**3. The Models & The Main Results.** In this section we present the dynamic algorithms, the accompanying assumptions and we will conclude by declaring the main results of the paper.

**3.1. Flocking to Synchronization.** The following algorithm assumes the scenario that agents, in addition to the consensus averaging, attain an individual way of flying dictated by an internal dynamical behavior.

$$i \in [N] : \begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = g(t, u_i) + \sum_j a_{ij}(t, \mathbf{x})(u_j - u_i), & t \geq t_0 \\ u_i(t) = u_i^0, x_i(t) = x_i^0 & t = t_0 \end{cases} \quad (3)$$

An agent’s state is affected both by the state of its neighboring nodes and by an inherent dynamical process, independent of the coupling process. Hence in the alignment problem is not clear if the condition of Theorem 1.1 suffices to ensure convergence. It is our goal to reveal the interplay between the coupling forces of the consensus network, the initial configuration and the potential instability induced by the internal dynamics through a new stability condition.

The assumption of the internal dynamics is stated below.

**Assumption 3.1.** *For the function  $g = (g_1, \dots, g_m)$  in (3) the following conditions hold:*

- (1.)  $g \in C^1([t, \infty) \times \mathbb{R}^m, \mathbb{R}^m)$ , in addition both  $g$  and  $g'$  are functions that are uniformly bounded in both arguments,
- (2.)  $\frac{\partial g_k(u)}{\partial u^{(p)}} = 0$  for  $p \neq k$ .

The second condition stated above implies that  $g$  is, in fact, decomposed among the  $m$ -dimensions. This is admittedly a hard assumption on  $g$  the necessity of which will be explained in the §6. In the meantime, we are allowed to consider, the following form on  $g$ :

$$g(t, u) = (g_1(t, u^{(1)}), \dots, g_m(t, u^{(m)})). \quad (4)$$

Next, we will state the assumption on the connectivity weights  $a_{ij}$ :

**Assumption 3.2.** *The coupling weights  $a_{ij}$  belong to  $C^0([t_0, \infty) \times \mathbb{R}^{N \times m}, [0, \infty))$  and satisfy*

$$\inf_{t \geq t_0} a_{ij}(t, \mathbf{x}) \geq \psi(S(\mathbf{x}))$$

for  $\psi \in C^0([t_0, \infty), [0, \infty))$  that is non-negative and non-increasing.

Assumption 3.2 allows couplings  $a_{ij}$  to vanish, as  $\psi(\cdot)$  is not necessarily bounded from below. Following the spirit of (2) the further apart two agents are, the weaker their communication should be so that it may become arbitrarily small.

For the statement of the first result we are in need of some additional notation. Set

$$\tilde{g} = \sup_{t \geq t_0} \max_l \max_{y_a, y_b \in \mathcal{U}_l} \int_0^1 g'_l(t, qy_a + (1-q)y_b) dq$$

where  $\mathcal{U}_l = [\min_i u_i^{(l)}(t_0), \max_i u_i^{(l)}(t_0)]$  and  $g'_l = \frac{\partial g_l(t, z)}{\partial z}$  that in view of Assumption 3.1 is bounded.

**Theorem 3.3.** *Consider the initial value problem (3) with Assumptions 3.1 and 3.2 to hold and its maximal solution  $(\mathbf{x}(t), \mathbf{u}(t))$ ,  $t \in [t_0, T)$ . If there exists  $r^* > 0$  such that*

$$S(\mathbf{u}^0) < \int_{S(\mathbf{x}^0)}^{r^*} (N\psi(r) - \tilde{g}) dr \text{ and } N\psi(r^*) > \tilde{g}, \quad (5)$$

then  $T = \infty$  whereas the solution satisfies

$$S(\mathbf{u}(t)) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ and } \sup_{t \geq t_0} S(\mathbf{x}(t)) < \infty.$$

**3.2. Collision Avoidance.** The second dynamic model we will study rolls back to the classic consensus problem and convergence to a common constant value but this should occur with agents staying at a minimum distance. The protocol we propose is

$$i \in [N] : \begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = \sum_j (a_{ij} + b_{ij})(u_j - u_i) \end{cases} \quad (6)$$

where

$$b_{ij} = -f_{ij}(\|x_{i,j}\|^2) \langle x_{i,j}, u_{i,j} \rangle / S(\mathbf{u})$$

models a well-defined, yet not-necessarily coupling, term. The functions  $f_{ij}$  are repelling forces that can be appropriately constructed so as to keep the agents at a relative distance.

**Assumption 3.4.** *For any  $i \neq j$ ,  $f_{ij}$  is any function  $f \in C^0((d_0, \infty), [0, \infty))$  satisfying for all  $d_1 > d_0$*

$$\int_{d_0}^{d_1} f(r) dr = \infty \text{ and } \int_{d_1}^{d_0} f(r) dr < \infty.$$

A simple example of repelling function (also to be used in §5) is given for any  $\varepsilon > 1$  by  $f(r) = (r - d_0)^{-\varepsilon}$ . For more examples we refer to [2]. Given the state-dependent type of couplings, what is the sufficient condition for flocking? One should expect a formula that connects the coupling strength with the repelling functions  $f_{ij}$ .

**Theorem 3.5.** <sup>5</sup> Consider the initial value problem (6) with Assumptions 3.2 and 3.4 to hold and its maximal solution  $(\mathbf{x}(t), \mathbf{u}(t))$  for  $t \in [t_0, T)$ . If for all  $i \neq j$  we have  $\|x_i^0 - x_j^0\| > d_0$  and

$$\frac{S(\mathbf{u}^0)}{N} < \int_{S(\mathbf{x}^0)}^{\infty} \psi(r) dr - \frac{1}{2} \max_{i \neq j} \int_{\|x_i^0 - x_j^0\|^2}^{\infty} f_{ij}(s) ds. \quad (7)$$

Then:

1.  $T = \infty$ ,
2.  $\|x_i(t) - x_j(t)\| > d_0$ , for all  $t \geq t_0$ ,
3. the solution satisfies  $S(\mathbf{u}(t)) \rightarrow 0$  as  $t \rightarrow \infty$  and  $\sup_t S(\mathbf{x}(t)) < \infty$ .

**4. Analysis.** This section is devoted to the proofs of Theorems 3.3 and 3.5. For the reader's convenience we break the proofs in instructive and gradual steps:

#### 4.1. Proof of Theorem 3.3.

Differential Inequality. At first we pick  $n = n_{t_0, T}$  large enough so that

$$n > \sup_{t \in [t_0, T)} \max_{i, i' \in [N]} \max_l \left\{ d_i(t) + \psi(S(\mathbf{x}(t))) - \int_0^1 g'_l(t, qu_i^{(l)}(t) + (1-q)u_{i'}^{(l)}(t)) dq \right\}$$

Such  $n$  exists in view of the Assumptions 3.1 and 3.2 and it is remarked that it is independent of  $t$ . Let  $(\mathbf{x}, \mathbf{u})$  be the maximal solution of (3). This is defined in an interval of type  $[t_0, T)$  for some  $t_0 < T \leq \infty$ . Next we re-write the vector equation in (3) as follows

$$\begin{aligned} \dot{u}_i &= -nu_i + (n - d_i)u_i + \sum_j a_{ij}u_j + g(t, u_i) \Leftrightarrow \\ e^{-nt} \frac{d}{dt}(e^{nt}u_i) &= (n - d_i)u_i + \sum_j a_{ij}u_j + g(t, u_i) \end{aligned}$$

where the differentiation is assumed to hold for  $t \in [t_0, T)$ . Pick  $i, i' \in [N]$  and  $l = \{1, \dots, m\}$ . From the above equation we take the difference:

$$e^{-nt} \frac{d}{dt}(e^{nt}(u_i^{(l)} - u_{i'}^{(l)})) = \sum_j \tilde{a}_{ij}^{(l)}(t)u_j^{(l)} - \sum_j \tilde{a}_{i'j}^{(l)}(t)u_j^{(l)}$$

where

$$\tilde{a}_{ij}^{(l)}(t) := \begin{cases} n - d_i(t) + \int_0^1 g'_l(t, qu_i^{(l)}(t) + (1-q)u_{i'}^{(l)}(t)) dq, & j = i \\ a_{ij}(\mathbf{x}(t)), & j \neq i, \end{cases}$$

or equivalently

$$e^{-nt} \frac{d}{dt}(e^{nt}(u_i^{(l)} - u_{i'}^{(l)})) = \sum_j w_j^{(l)}(t)u_j^{(l)}$$

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<sup>5</sup>The authors acknowledge a typo in the initial configuration condition that has already appeared in the proceedings version of this work [22]. Theorem 3.5 presented here is the correct version.

for  $w_j^{(l)}(t) = \tilde{a}_{ij}^{(l)}(t) - \tilde{a}_{i'j}^{(l)}(t)$ . The index for which  $w_j^{(l)}(t) > 0$  is denoted by  $j^+$  and the index for which  $w_j^{(l)}(t) \leq 0$  is denoted by  $j^-$ . It is of crucial important to note that  $\tilde{a}_{ij}^{(l)}(t)$  as they were defined imply  $\sum_j w_j^{(l)}(t) \equiv 0$ . Set

$$\begin{aligned}\theta(t) &= \sum_{j^+} w_{j^+}^{(l)}(t) = \sum_{j^+} |w_{j^+}^{(l)}(t)| = - \sum_{j^-} w_{j^-}^{(l)}(t) = \sum_{j^-} |w_{j^-}^{(l)}(t)| = \frac{1}{2} \sum_j |w_j^{(l)}(t)| \\ &= \frac{1}{2} \sum_j |\tilde{a}_{ij}^{(l)}(t) - \tilde{a}_{i'j}^{(l)}(t)|\end{aligned}$$

Then for  $t \in [t_0, T)$

$$e^{-nt} \frac{d}{dt} (e^{nt} (u_i - u_{i'})) = \theta(t) \left( \frac{\sum_{j^+} |w_{j^+}^{(l)}(t)| u_{j^+}}{\theta(t)} - \frac{\sum_{j^-} |w_{j^-}^{(l)}(t)| u_{j^-}}{\theta(t)} \right) \leq \theta(t) S(\mathbf{u})$$

But from the identity  $|x - y| = x + y - 2 \min\{x, y\}$  we deduce

$$\theta = n + \int_0^1 g'_l(t, qu_i + (1-q)u_{i'}) dq - \sum_k \min\{\tilde{a}_{ik}, \tilde{a}_{i'k}\}.$$

Now the choice of  $n$  we made in the beginning of the proof implies that the summation of the minima over  $k$  cannot include  $\tilde{a}_{ii}^{(l)}(t)$  or  $\tilde{a}_{i'i'}^{(l)}(t)$ , hence we are left with the off-diagonal elements and the following upper bound on  $\theta(t)$ :

$$\theta(t) \leq n - N\psi(S(\mathbf{x})) + \max_{u_i, u_{i'}} \int_0^1 g'_l(t, qu_i + (1-q)u_{i'}) dq.$$

Finally, for  $i, i'$  and  $l$  that maximize  $u_i^{(l)} - u_{i'}^{(l)}$ , we have

$$\begin{aligned}\frac{d}{dt} S(\mathbf{u}) &= \frac{d}{dt} [e^{-nt} S(e^{nt} \mathbf{u})] = -nS(\mathbf{u}) + e^{-nt} \frac{d}{dt} S(e^{nt} \mathbf{u}) \\ &\leq -N\psi(S(\mathbf{x})) S(\mathbf{u}) + \bar{g} S(\mathbf{u}).\end{aligned}\tag{8}$$

**Asymptotic Flocking.** In order to prove exponential convergence to zero (hence synchronized flocking) we need to show at first that  $T = \infty$  (so that  $n$  can be chosen independent of  $T$ ) as well as that

$$N\psi\left(\sup_{t \geq t_0} S(\mathbf{x}(t))\right) > \bar{g}$$

where we note that  $\bar{f}$  is a function of the initial configuration of the velocities. We will explain now how (5) guarantees all the above. Consider the functional

$$\mathcal{V}(\mathbf{x}, \mathbf{u}) = S(\mathbf{u}) + \int_0^{S(\mathbf{x})} (N\psi(r) - \bar{g}) dr\tag{9}$$

and evaluate it at the solution  $(\mathbf{x}(t), \mathbf{u}(t)), t \in [t_0, T)$  with  $\mathcal{V}(t) = \mathcal{V}(\mathbf{x}(t), \mathbf{u}(t))$ . From (5) there exists  $t_1 > t_0$  such that for  $t \in [t_0, t_1)$

$$\begin{aligned}\frac{d}{dt} \mathcal{V}(t) &\leq 0 \Rightarrow \mathcal{V}(t) \leq \mathcal{V}(t_0) \Rightarrow \\ S(\mathbf{u}(t)) + \int_0^{S(\mathbf{x}(t))} (N\psi(r) - \bar{g}) dr &\leq S(\mathbf{u}^0) + \int_0^{S(\mathbf{x}^0)} (N\psi(r) - \bar{g}) dr\end{aligned}$$

and obviously

$$\int_0^{S(\mathbf{x}(t))} (N\psi(r) - \bar{g}) dr \leq S(\mathbf{u}^0) + \int_0^{S(\mathbf{x}^0)} (N\psi(r) - \bar{g}) dr$$

Once more the initial configuration (5) implies the existence of  $s^* < r^*$  such that

$$S(\mathbf{u}^0) = \int_{S(\mathbf{x}^0)}^{s^*} (N\psi(r) - \bar{g}) dr \text{ and } N\psi(s^*) > \bar{g}$$

in view of the monotonicity of  $\psi$  in Assumption 3.2. Then

$$\int_0^{S(\mathbf{x}(t))} (N\psi(r) - \bar{g}) dr \leq \int_0^{s^*} (N\psi(r) - \bar{g}) dr$$

so

$$\int_{S(\mathbf{x}(t))}^{s^*} (N\psi(r) - \bar{g}) dr > 0$$

The last condition implies that  $S(\mathbf{x}(t)) \leq s^*$  for  $t < t_1$ . Since no assumption was taken on  $t_1$ , the monotonicity of  $\psi$  yields that  $t_1 = \infty$ , i.e.  $T = \infty$ . The differential inequality (8) is then strictly negative, and  $\mathbf{u}$  is trapped in the initial configuration area. The number  $n$  is then a number independent of the  $T$  it remains well-defined and constant for all  $t \geq t_0$  concluding the proof.

**4.2. Proof of Theorem 3.5.** The proof of Theorem 3.5 follows partly the steps of the proof of Theorem 3.3 and partly the arguments developed in [2].

Differential Inequality I. We turn to the second equation of (6). The maximal solution  $(\mathbf{x}(t), \mathbf{u}(t))$  is defined in  $[t_0, T)$  and it satisfies

$$\dot{u}_i = -nu_i + \alpha_{ii}(t)u_i + \sum_j \alpha_{ij}(t)u_j$$

for

$$\alpha_{ij}(t) := \begin{cases} \tilde{a}_{ij}(t) + \tilde{b}_{ij}(t), & i \neq j \\ \tilde{a}_{ii}(t) + \tilde{b}_{ii}(t), & i = j. \end{cases}$$

where  $\tilde{a}_{ij}(t) := a_{ij}(\mathbf{x}(t))$ ,  $\tilde{a}_{ii}(t) := \frac{n}{2} - \sum_{j \neq i} \tilde{a}_{ij}(t)$  and  $\tilde{b}_{ij}(t) := b_{ij}(\mathbf{x}(t), \mathbf{u}(t))$ ,  $\tilde{b}_{ii}(t) := \frac{n}{2} - \sum_{j \neq i} \tilde{b}_{ij}(t)$ . Pick  $i, i' \in [N]$  and  $l = 1, \dots, m$  as in the proof of the previous section and consider the difference

$$\dot{u}_i^{(l)} - \dot{u}_{i'}^{(l)} = \sum_j w_j(t) u_j^{(l)}$$

for  $w_j(t) = \alpha_{ij}(t) - \alpha_{i'j}(t)$ . Let  $j'$  denote the indices  $j$  for which  $w_j(t) > 0$  and  $j''$  the indices for which  $w_j(t) \leq 0$ . Then

$$\dot{u}_i^{(l)} - \dot{u}_{i'}^{(l)} = \sum_j |w_j(t)| u_j^{(l)} - \sum_{j''} |w_{j''}(t)| u_{j''}^{(l)}.$$



Also we note that  $\sum_j w_j(t) \equiv 0$  so for  $\theta(t) = \sum_{j'} w_{j'}(t)$  we have

$$\begin{aligned}\theta &= \sum_{j'} |w_{j'}(t)| = - \sum_{j''} w_{j''}(t) = \sum_{j''} |w_{j''}(t)| = \frac{1}{2} \sum_j |w_j(t)| \\ &= \frac{1}{2} \sum_j |\alpha_{ij} - \alpha_{i'j}| \leq \frac{1}{2} \sum_j |a_{ij} - a_{i'j}| + \frac{1}{2} \sum_j |b_{ij} - b_{i'j}| \\ &= n - \sum_{j \neq i} \min\{a_{ij}, a_{i'j}\} - \sum_{j \neq i} \min\{b_{ij}, b_{i'j}\}\end{aligned}$$

in view of the identity  $2 \min\{x, y\} = x + y - |x - y|$  and  $n$  taken sufficiently large. Consequently  $\theta(t)$  is upper bounded as follows

$$\theta(t) \leq n - \rho(t) - \sigma(t)$$

where  $\rho(t) := \min_{i, i'} \sum_j \min\{a_{ij}(t), a_{i'j}(t)\}$  and

$$\sigma(t) := \min_{i, i'} \sum_j \min\{b_{ij}(t), b_{i'j}(t)\}.$$

Finally,

$$\begin{aligned}\dot{u}_i^{(l)} - \dot{u}_{i'}^{(l)} &\leq \theta \left( \frac{\sum_{j'} |w_{j'}| u_{j'}}{\sum_{j'} |w_{j'}|} - \frac{\sum_{j''} |w_{j''}| u_{j''}}{\sum_{j''} |w_{j''}|} \right) \\ &\leq (n - \rho(t)) S(\mathbf{u}) - \sigma(t) S(\mathbf{u})\end{aligned}$$

Following the proof of Theorem 3.3 we derive the differential inequality

$$\frac{d}{dt} S(\mathbf{u}) \leq -\rho(t) S(\mathbf{u}) - \sigma(t) S(\mathbf{u}). \quad (10)$$

The first step is concluded by noting that  $a_{ij}$  satisfy Assumption 3.2 hence  $\rho(t)$  is bounded from below as

$$\rho(t) \geq N\psi(S(\mathbf{x}(t))). \quad (11)$$

Differential Inequality II. On the other hand,  $\sigma(t)$  may be positive or negative. We observe that  $\tilde{b}_{ij}(t)$  can be written as

$$\tilde{b}_{ij}(t) = \frac{1}{2S(\mathbf{u})} \frac{d}{dt} \int_{\|x_{i,j}\|^2}^{\infty} f_{ij}(s) ds.$$

Let now  $h, h' \in [N]$  be the (possibly  $t$ -dependent but piecewise constant) indices that minimize  $\tilde{b}_{ij}(t)$ . From the monotonic character of  $f_{ij}$ ,  $h, h'$  are the ones that correspond to the two agents that they are both closest to each other and at the same time they approach each other. If at some interval of time such a pair does not exist then all agents must diverge from each other and then one can pick a pair of agents that diverge from each other and minimizes the aforementioned quantity. Then

$$\sigma(t) \geq \frac{N}{2S(\mathbf{u})} \frac{d}{dt} \int_{\|x_{h,h'}\|^2}^{\infty} f_{hh'}(s) ds. \quad (12)$$

Combine (10) with (11) and (12) to conclude

$$\frac{d}{dt} S(\mathbf{u}) \leq -N\psi(S(\mathbf{x}(t))) S(\mathbf{u}) - \frac{N}{2} \frac{d}{dt} \int_{\|x_{h,h'}\|^2}^{\infty} f_{hh'}(s) ds \quad (13)$$

Collision Avoidance. We explain now that (13) guarantees that any configuration with initial data as in 3.5 is guaranteed to generate solutions that satisfy the non-collision condition for all  $t \geq t_0$ . For the sake of convenience

$$\mathcal{E}(\mathbf{x}, \mathbf{u}) = S(\mathbf{u}) + \frac{N}{2} \int_{||x_{h,h'}||^2}^{\infty} f_{hh'}(s) ds$$

where  $h$  and  $h'$  are the agents with the characteristics described in the previous step. The need for such consideration allows to rewrite (10)

$$\frac{d}{dt}\mathcal{E}(t) \leq -N\psi(S(\mathbf{x}(t)))S(\mathbf{u}) < 0 \quad (14)$$

for  $t \in [t_0, T)$ . Then

$$\mathcal{E}(t) \leq \mathcal{E}(t_0) \Rightarrow \int_{||x_h(t)-x_{h'}(t)||^2}^{\infty} f_{hh'}(s) ds \leq \mathcal{E}(t_0) < \infty$$

The nature of  $h$  and  $h'$  as discussed above implies that  $||x_{i,j}|| > d_0$  for all  $i$  and  $j$  in  $[t_0, T)$ .

Existence in the Large. From (14) we also deduce that  $S(\mathbf{u}(t)) \leq \mathcal{E}(t_0) < \infty$  hence  $S(\mathbf{x}(t)) \leq S(\mathbf{x}^0) + T\mathcal{E}(t_0)$  and the solution lies for  $[t_0, T)$  in

$$\Omega = \{(\mathbf{x}, \mathbf{u}) : S(\mathbf{x}) \leq T, ||x_{i,j}|| \geq d, i \neq j, S(\mathbf{u}) \leq \mathcal{E}(t_0)\}$$

but this a compact subset of

$$\{(\mathbf{x}, \mathbf{u}) : S(\mathbf{x}) \leq T, ||x_{i,j}|| > d_0, i \neq j\}.$$

The fundamentals in the theory of differential equations assure, however, that this cannot occur if  $T < \infty$  (see for example [9]) hence it is ensured that the solution is eventually defined for all  $t \geq t_0$ .

Bounded Flock. We will show now that the initial settings we consider in the statement of the Theorem keeps all agents in finite relative distance. For this we integrate (14) from  $t_0$  to  $t$  to obtain

$$\begin{aligned} \mathcal{E}(t) - \mathcal{E}(t_0) &\leq - \int_{t_0}^t n\psi(S(x(s)))S(u(s)) ds \Rightarrow \\ \mathcal{E}(t_0) &\geq \int_{t_0}^t n\psi(S(x(s)))S(u(s)) ds. \end{aligned} \quad (15)$$

If we set  $v = S(x(s))$  we observe that  $\frac{dv}{ds} \leq S(u(s))$  and deduce

$$\int_{S(x^0)}^{S(x)} n\psi(v) dv \leq \mathcal{E}(t_0).$$

Had the flock been dissolved, there should be a sequence  $\{t_i\}_{i \geq 0}$  with  $t_i \rightarrow \infty$  as  $i \rightarrow \infty$ , so that  $S(x(t_i)) \rightarrow \infty$  as  $t_i \rightarrow \infty$ . This would mean that

$$\int_{S(x^0)}^{\infty} n\psi(v) dv \leq \mathcal{E}(t_0)$$

that is incompatible with the initial configuration (7). Thus

$$\sup_{t \geq t_0} S(x(t)) < \infty, \quad (16)$$

i.e. the flock remains connected and bounded.

Convergence to flocking. It only remains to show that  $S(\mathbf{u}(t)) \rightarrow 0$  as  $t \rightarrow \infty$ . At first we combine (15) and (16) to conclude

$$\int_{t_0}^{\infty} S(u(s)) ds < \infty.$$

Moreover, we used (14) to show that  $\sup_{t \geq t_0} S(u(t)) < \mathcal{E}(t_0)$  and this implies that  $|S(u(t))|$  is uniformly bounded. It only remains to show that  $S(u(t))$  is uniformly continuous so that Barbalat's Lemma applies to  $k(t) := \int_{t_0}^t S(u(s)) ds$  to conclude

$$\frac{d}{dt}k(t) = S(u(t)) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

The last condition is shown by direct application of the definition of uniform continuity: For any  $t_1, t_2 \geq t_0$  close to each other, there exist  $i, i' \in [n]$  and  $l \in [r]$  such that

$$\begin{aligned} |S(u(t_1)) - S(u(t_2))| &= |(u_i^{(l)}(t_1) - u_{i'}^{(l)}(t_1)) - (u_i^{(l)}(t_2) - u_{i'}^{(l)}(t_2))| \\ &\leq 2 \max_{h \in \{i, i'\}} |u_h^{(l)}(t_1) - u_h^{(l)}(t_2)| \\ &\leq 2 \max_{h \in \{i, i'\}} |\dot{u}_h^{(l)}(t^*)| |t_1 - t_2| \end{aligned}$$

Since  $\dot{u}_h^{(l)}(t)$  satisfied (6), its absolute value is bounded above by finite number of terms  $|w_{ij}| \leq \bar{w}$ ,  $|b_{ij}(x, u)| \leq \max_{ij} f_{ij}(\underline{d}^2)(\sup_t S(x(t)))^2 \mathcal{E}(t_0)$  where  $\underline{d} = \inf_t \min_{i \neq j} |x_{ij}(t)| > d_0$  and  $|S(u(t))| \leq \mathcal{E}(t_0)$ , each of which is independent of time and the uniform continuity property holds true. Barbalat's lemma can then be applied concluding the proof.

**5. Simulation Examples.** In this section we will briefly present two simple examples. Due to space limitation we will apply Theorems 3.3 and 3.5 in an elementary but illustrative way. Exhaustive numerical investigations on the initial conditions formulas of the theorems are postponed for the extended version of this work. Both examples are among a network of  $N = 5$  agents with dynamics evolving on the plane ( $m = 2$ ). The coupling functions are assumed

$$a_{ij}(t, \mathbf{x}) = (1 + 0.9 \sin(t)) \frac{\psi(\|x_i - x_j\|)}{\sum_j \psi(\|x_i - x_j\|)}$$

and  $\psi(r) = \frac{1}{1+r^\varepsilon}$  for  $\varepsilon \geq 0$ . The initial time is taken  $t_0 = 0$ . All simulations were carried with the `ode23` routine in `MATLAB`.

**5.1. Example 1. Synchronization.** We assume the following internal dynamics

$$g(t, z_1, z_2) = (\sin(t) \cos(z_1), \cos(t/3) \sin(z_2)).$$

Standard ODE arguments yield that the forward limit set of the synchronized state is non-empty and compact. We illustrate the case  $\varepsilon = 1$  which implies that the coupling strength is strong enough for arbitrary initial data and the case  $\varepsilon = 1.5$  of a weak coupling strength. Then appropriate initial data violate the condition of Theorem 3.3. In Figs. 1 and 2 we provide simulations where the condition of the Theorem is satisfied and the condition of the Theorem is violated, respectively.

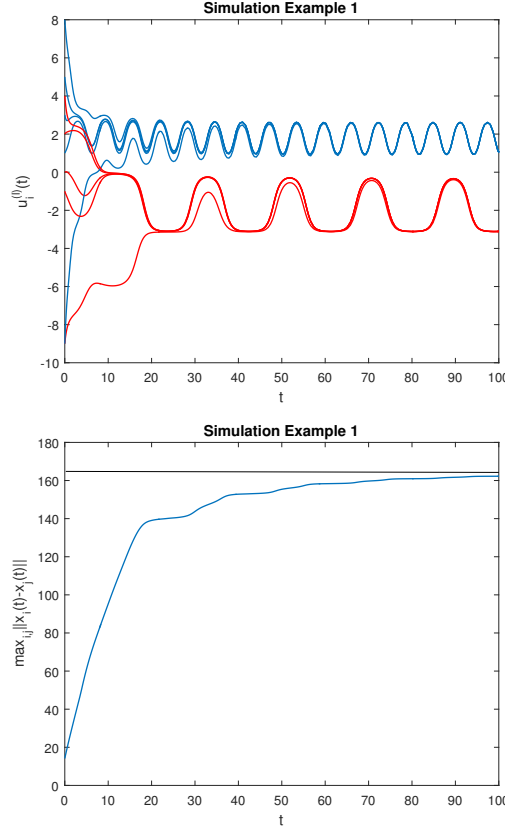


FIGURE 1. Simulations with strong coupling  $\varepsilon = 1$  and the condition of Theorem 3.3 applies for any initialization of the initial states. The synchronization of agents' velocities occurs in the two dimensions, around the nominal orbits that are solutions of  $\dot{v}_1 = g_1(t, v_1)$  and  $\dot{v}_2 = g_2(t, v_2)$ .

5.2. **Example 2. Collision Avoidance.** The repelling functions are taken

$$f_{ij}(r) = \frac{K_{ij}}{(r - r_0)^\varphi}$$

for numbers  $K_{ij}$  arbitrarily chosen from  $(1, 2)$  and  $\varphi = 1.5$  and  $r_0 = 0.25$ . We set  $\varepsilon = 1$  so that the condition of Theorem 3.5 is clearly satisfied. See Fig. 3 for the simulation results.

6. **Discussion.** In the present paper we addressed two variations of the standard non-linear flocking algorithm of Cucker-Smale type with asymmetric system parameters. We exploited a standard tool from non-negative matrix theory known as the contraction coefficient. A proper modification of these concepts enable us to study two seemingly different extension of flocking networks. The reader should observe the striking similarity in the analysis of the synchronization and the collision avoidance flocking algorithms. Both systems include the consensus-based stabilizing term and a potentially destabilizing term.

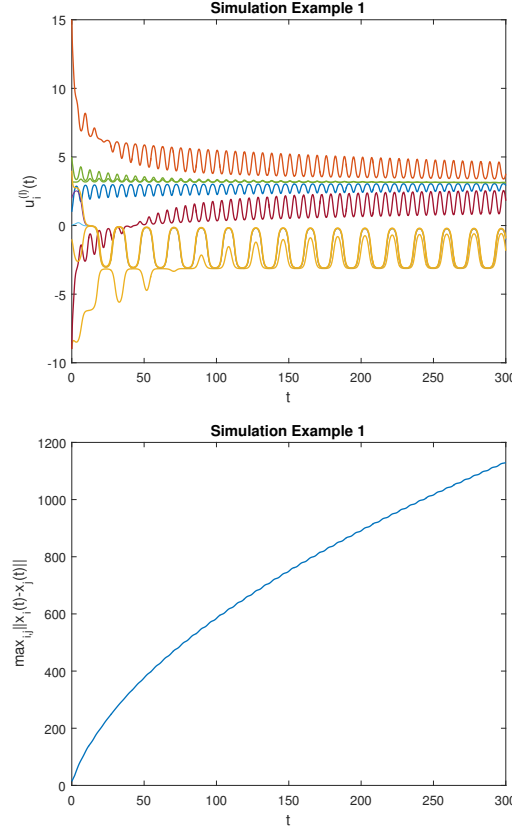


FIGURE 2. Simulations with weak coupling  $\varepsilon = 1.5$  and the condition of Theorem 3.3 does not apply for large velocity initialization. The spread  $S(\mathbf{u}^0)$  is too large and the first condition is violated. There is an apparent failure of velocity synchronization and a subsequent divergence of the relative distance that results in the dissolution of the flock.

In the first example instability occurs from the existence of multiple forward limit sets of the internal dynamics and the large enough deviation in the initial conditions. In such a case we seek a formula on the initial settings that explains how strong a coupling must be in order to prevail on the induced instability and drive the network to flocking condition. Indeed observe that if  $\dot{z} = g(t, z)$  is a dynamical system with a globally attracting limit set then  $g'$  should be negative evaluated at each point of the state space. Then  $\bar{g}$  is negative and it also contributes to the stability of the network. This is easily deduced from the condition of Theorem 3.3 which in such case yields a larger right hand side.

In the second example, the network can be destabilized by a collision avoidance term. Agents that approach too close to each other cause instability that can affect the rest of the network as Figure 3 clearly shows. This instability demands a strong coupling to hold the flock together. This is illustrated in Theorem 3.5 and it is a condition strikingly similar to a similar network introduced in [2]. However our

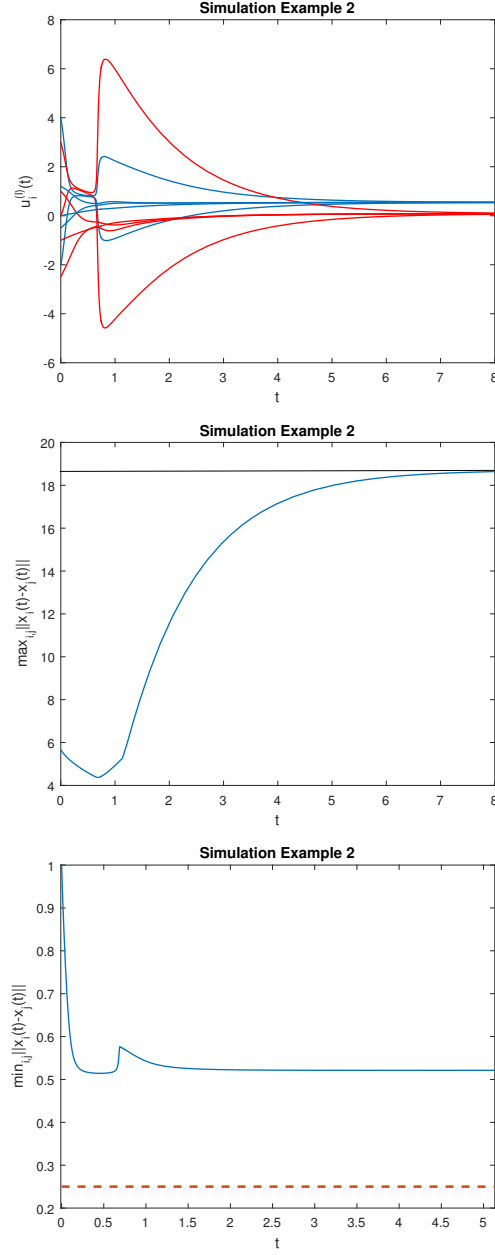


FIGURE 3. Simulation Example 2. Flocking with collision avoidance. We observe the velocity alignment in each dimension as well as the monitoring of the maximum and the minimum relative distance over the agents with the corresponding bounds.

model is free of any symmetry assumptions on the coupling and the repelling terms that the authors there were obliged to assume following algebraic graph theory techniques.

Further inspection (both numerical and theoretical) of the initial settings formulas, as well as modeling of weaker network topologies (not necessarily all-to-all communication) are problems for future research.

The main drawback of this work is the strong Assumption 3.1 on the decomposition of  $g$ . Following the steps of the proof of Theorem 3.3 one can see that if condition (2) of Assumption 3.1 is violated then our analysis ceases to hold. This issue is also mentioned in [21]. It is still not clear how one should proceed in adapting the concept of the coefficient of ergodicity to handle systems with more general nominal dynamics in many dimensions. We consider this also as a very interesting open problem for future research.

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